Congruences between abelian pseudomeasures, II

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In this paper we generalize the main result of [RW7]. As there, K is a totally real number field (finite over $\mathbb{Q}$), p a fixed prime number, and S a fixed finite set of non-archimedean primes of K containing all primes above p. Let K_S denote the maximal abelian extension of K which is unramified (at all non-archimedean primes) outside S and set $G_S = G(K_S/K)$. Serre's pseudomeasure $\lambda_K = \lambda_{K,S}$ has the property that $(1-g)\lambda_K$ is in the completed group ring $\mathbb{Z}_p[[G_S]]$ for all $g \in G_S$ [Se].

Let L be a totally real Galois extension of K of p-power degree with group $\Sigma = G(L/K)$. We require that the finite set S contains all the primes of K which ramify in L. Letting F run through the intermediate fields of L/K, we denote by

F_S the maximal abelian extension of F unramified outside the primes of F above S,

F_S^+ its maximal totally real subfield (hence $F \subset F_S^+$),

$\lambda_{F,S}$ Serre's pseudomeasure with respect to F and S,

$H_S = G(L_S/L)$, $H_S^+ = G(L_S^+/L)$.

Note that $G(L_S/F)^{\mathrm{ab}} = G(F_S/F)$ and, for $F \subseteq F'$, that $G(L_S/F')$ is an open subgroup of $G(L_S/F)$. This yields the transfer map $G(F_S/F) \to G(F'_S/F')$, and in particular,

$$\mathrm{ver}_K^F : G_S \to G(F_S/F)\,,\ \mathrm{ver}_F^L : G(F_S/F) \to H_S\,,\ \mathrm{ver}_K^L = \mathrm{ver}_F^L \circ \mathrm{ver}_K^F : G_S \to H_S\,.$$

In order to state the new main result we still need to recall the definition of the Möbius function $\mu = \mu_\Sigma$ of the poset of subgroups of the finite group Σ. It is defined by

$$\mu(1) = 1\,,\ \mu(\Sigma') = -\sum_{1 \le \Sigma'' < \Sigma'} \mu(\Sigma'') \text{ for } 1 \ne \Sigma' \le \Sigma\,.$$

For $K \subseteq F \subseteq L$ and $g \in G_S$ set $\tilde\lambda_F = 2^{-[F:\mathbb{Q}]}\lambda_{F,S}$ and $\tilde\lambda_{g_F} = (1-g_F)\tilde\lambda_F$, where $g_F = \mathrm{ver}_K^F g$. Moreover, denote the Galois group of the cyclotomic $\mathbb{Z}_p$-extension F_∞/F by Γ_F.

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Theorem. *There exist $g \in G_S$ so that g_F has image $\neq 1$ in Γ_F for $K \subseteq F \subseteq L$ and the image of*

$$\sum_{K\subseteq F\subseteq L} \mu_\Sigma(G(L/F))\mathrm{ver}_F^L(\tilde{\lambda}_{g_F})\,,$$

under $\mathbb{Z}_p[[H_S]] \to \mathbb{Z}_p[[H_S^+]]$, is in the trace ideal $\mathrm{tr}_\Sigma(\mathbb{Z}_p[[H_S^+]]$ of $\mathbb{Z}_p[[H_S^+]]^\Sigma$.

The theorem and its proof generalize [RW7], where Σ had order p, and therefore are an application of the methods of [DR] in the language of [Se]. The precise formulation, however, is dictated by the needs of equivariant Iwasawa theory [RW2,p.564].

More precisely, there one has a profinite extension $\mathbf{K}/K$ of totally real fields with Galois group G, with $K/\mathbb{Q}$ finite and $\mathbf{K}$ containing the cyclotomic $\mathbb{Z}_p$-extension K_∞ of K, such that $[\mathbf{K} : K_\infty] < \infty$. In this situation, the so-called *'main conjecture' of equivariant Iwasawa theory* [1] concerns the *Iwasawa L-function* $L_{\mathbf{K}/K,S\cup S_\infty}$, which is built from the $S \cup S_\infty$-truncated [2] p-adic Artin L-functions $L_{p,K,S\cup S_\infty}(s,\chi)$ attached to the $\mathbb{Q}_p^c$-irreducible characters χ of G with open kernel (see loc.cit. – but with $\mathbf{K}$, K, p denoted K, k, l, respectively). Assuming, for odd p, Iwasawa's conjecture that the μ-invariant for $\mathbf{K}/K$ vanishes, the 'main conjecture', up to its uniqueness assertion, is equivalent to

$$L_{\mathbf{K}/K,S\cup S_\infty} \quad \text{is in} \quad \mathrm{Det}\,(K_1(\Lambda_\wedge G))\,,$$

where $\Lambda_\wedge G$ is the completion of the localization of $\mathbb{Z}_p[[G]]$ obtained by inverting all elements which are regular modulo p (see [RW3, Theorem A] [3]).

Now assuming that G is a p-elementary group, i.e., a direct product of a finite cyclic group of order prime to p and a pro-p group, and picking an abelian normal open subgroup A of G with factor group $\Sigma = G/A$ a p-group, the *Möbius-Wall congruence*

(MW) $$\textstyle\sum_{A\leq U\leq G} \mu_\Sigma(U/A)\mathrm{ver}_U^A(\mathrm{res}_G^U y) \equiv 0 \mod \mathrm{tr}_\Sigma(\Lambda_\wedge A)$$

holds for all units $y \in \Lambda_\wedge G$. (MW) is in [RW10, §2].

If the 'main conjecture' were true, so $\mathrm{Det}(y) = L_{\mathbf{K}/K,S\cup S_\infty}$ with a $y \in K_1(\Lambda_\wedge G)$, then on plugging y into (MW) we would obtain

$$\sum_{A\leq U\leq G} \mu_\Sigma(U/A)\mathrm{ver}_{U^{\mathrm{ab}}}^A(\lambda_{U^{\mathrm{ab}}}) \equiv 0 \mod \mathrm{tr}_\Sigma(\Lambda_\wedge A)\,.$$

And this *abelian pseudomeasure congruence* turns out to be a consequence of the Theorem stated above. This is addressed in detail in [RW10], where the 'main conjecture' for arbitrary extensions $\mathbf{K}/K$, up to its uniqueness assertion, is verified.

The organization of the paper parallels [RW7] to the extent that, with two exceptions [4], the numbering and, mutatis mutandis, statements of the lemmas and propositions persist, with

[1] see also e.g. [CFKSV; FK; Ha1,2; K; Ka; RW3,7,8]
[2] S_∞ is the set of archimedean primes of k
[3] for $p = 2$, compare [RoW]
[4] Propositions 5 and 9

proofs omitted if they are essentially already in [RW7]. The new ingredient is the identification of the congruence (4$\star$) of Proposition 4 as the difference of constant terms of q-expansions at two cusps of a Hilbert modular form of Eisenstein type. This modular form is exhibited in §2 and then studied via the q-expansion principle of Deligne and Ribet; the hypothesis of Proposition 4 is deduced, in Proposition 9, from a property of Möbius coefficients in [HIÖ]. The proof, in §3, of the main result is then a computation of constant terms of q-expansions at the "special cusps" of Proposition 5.

Our special cusps are a simple device to avoid comparing constant coefficients of q-expansions of F and $F_{|k}U_\beta$ at *arbitrary* cusps in Lemma 6. Having overlooked the need for this comparison in [RW7] implies that we now have its Theorem only for special $g = g_K \in G_S$. However, the present theorem is better suited for equivariant Iwasawa theory; see the Remark in §3.

1. A sufficient condition for a pseudomeasure congruence

For a coset x of an open subgroup U of $G(F_S/F)$ set $\delta^{(x)}(g) = 1$ or 0 according as $g \in x$ or not. Then, for even integers $k \geq 1$, define $\tilde{\zeta}_F(1-k, \delta^{(x)}) = 2^{-[F:\mathbb{Q}]}\zeta_{F,S}(1-k,\delta^{(x)}) \in \mathbb{Q}$ to be $2^{-[F:\mathbb{Q}]}$ times the value at $1-k$ of the partial ζ-function for the set of integral ideals $\mathfrak{a}$ of F prime to S with Artin symbol $(\mathfrak{a}, F_S/F)$ in x. Note that the definition of $\tilde{\zeta}_F(1-k,\delta^{(x)})$ extends linearly to locally constant functions ε on $G(F_S/F)$ with values in a $\mathbb{Q}$-vector space and gives values $\tilde{\zeta}_F(1-k,\varepsilon)$ in that vector space, as usual.

Let $\mathcal{N} = \mathcal{N}_{F,p} : G(F_S/F) \to \mathbb{Z}_p{}^\times$ be that continuous character whose value on $(\mathfrak{a}, F_S/F)$ for an integral ideal $\mathfrak{a}$ of F prime to S is its absolute norm $\mathcal{N}_F\mathfrak{a}$. For $g \in G(F_S/F)$, $k \geq 1$ and ε a locally constant $\mathbb{Q}_p$-valued function on $G(F_S/F)$ we define, following [DR],

$$\tilde{\Delta}_g(1-k,\varepsilon) = \tilde{\zeta}_F(1-k,\varepsilon) - \mathcal{N}(g)^k\tilde{\zeta}_F(1-k,\varepsilon_g) \in \mathbb{Q}_p\,,$$

where $\varepsilon_g(g') = \varepsilon(gg')$ for $g' \in G(F_S/F)$.

Theorem [(0.4) of [DR]]. *Let $\varepsilon_1, \varepsilon_2, \ldots$ be a finite sequence of locally constant functions $G(F_S/F) \to \mathbb{Q}_p$ so that $\sum_{k\geq 1}\varepsilon_k(g')(\mathcal{N}g')^{k-1} \in \mathbb{Z}_p$ for all $g' \in G(F_S/F)$. Then*

$$\sum_{k\geq 1}\tilde{\Delta}_g(1-k,\varepsilon_k) \in \mathbb{Z}_p \quad \text{for all } g \in G(F_S/F)\,.^5$$

Call an open subgroup U of $G(F_S/F)$ *admissible*, if $\mathcal{N}(U) \subset 1 + p\mathbb{Z}_p$, and define $m_F(U) \geq 1$ by $\mathcal{N}(U) = 1 + p^{m_F(U)}\mathbb{Z}_p$.

Lemma 1. *If U runs through the cofinal system of admissible open subgroups of $G(F_S/F)$, then $\mathbb{Z}_p[[G(F_S/F)]] = \varprojlim_U \mathbb{Z}_p[G(F_S/F)/U]/p^{m_F(U)}\,\mathbb{Z}_p[G(F_S/F)/U]$.*

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[5] For $p = 2$ compare [RW7,§5]

Proposition 2. *For $h \in G(F_S/F)$ there is a unique element $\tilde{\lambda}_h \in \mathbb{Z}_p[[G(F_S/F)]]$, independent of k, whose image in $\mathbb{Z}_p[G(F_S/F)/U]/p^{m_F(U)}$ is*

$$\sum_{x \in G(F_S/F)/U} \tilde{\Delta}_h(1-k, \delta^{(x)})\mathcal{N}(x)^{-k}x \mod p^{m_F(U)}\mathbb{Z}_p[G(F_S/F)/U]$$

for all admissible U, where $\mathcal{N}$ here also denotes the homomorphism $G(F_S/F)/U \to (\mathbb{Z}_p/p^{m(U)})^\times$ induced by our previous $\mathcal{N}$. Moreover, if $\tilde{\lambda}_F$ is $2^{-[F:\mathbb{Q}]}$ times the pseudomeasure of [Se], then $(1-h)\tilde{\lambda}_F = \tilde{\lambda}_h$.

Lemma 3. *(1) Let V be an admissible open subgroup of H_S. If $U \leq (\mathrm{ver}_F^L)^{-1}(V)$, then $m_F(U) \geq m_L(V) - e_F$ where $[L:F] = p^{e_F}$.*

(2) Let $s \in G(L_S/K)$ be an extension of $\sigma \in \Sigma$. Then $(F^\sigma)_S = (F_S)^s$ and, setting $g_F = \mathrm{ver}_K^F(g)$ for $g \in G_S$, $g_F^s = g_{F^\sigma}$. Moreover, $\mathrm{ver}_F^L(\tilde{\lambda}_{g_F})^s = \mathrm{ver}_{F^\sigma}^L(\tilde{\lambda}_{g_{F^\sigma}})$.

Statement (1) is due to $\mathcal{N}_L(\mathrm{ver}_F^L g) = \mathcal{N}_F(g)^{[L:F]}$ for $g \in G(F_S/F)$. For (2), the first claim follows from $(F_S)^s \supseteq F^s = F^\sigma$ and $G((F_S)^s/F^s) = G(F_S/F)^s$, the latter implying $(F_S)^s \subseteq (F^\sigma)_S$ and then $F_S \subseteq (F^\sigma)_S^{s^{-1}} \subseteq F_S$, hence equality everywhere.

The second claim is a direct consequence of the definition of group transfer 'ver'. Namely $\mathrm{ver}_K^F(g)$ is a certain product built with respect to coset representatives of $G(L_S/F)$ in $G(L_S/K)$, which s takes to coset representatives of $G(L_S/F^\sigma)$ in $G(L_S/K)$, whence the multiplicativity of s yields $\mathrm{ver}_K^F(g)^s = \mathrm{ver}_K^{F^\sigma}(g^s)$. But $g \in G_S = G(L_S/K)^{\mathrm{ab}}$ implies $g^s = g$.

Finally, concerning the last claimed equality, by Proposition 2 and $\mathcal{N}_{F^\sigma}(x^s) = \mathcal{N}_F(x)$ it suffices to show

$$\tilde{\Delta}_{g_{F^\sigma}}(1-k, \delta_{F^\sigma}^{(x^s)}) = \tilde{\Delta}_{g_F}(1-k, \delta_F^{(x)})$$

for x a coset of any admissible open $G(L/F)$-stable subgroup U of $G(F_S/F)$. Thus it suffices to show

$$\tilde{\zeta}_{F^\sigma}(1-k, \delta_{F^\sigma}^{(x^s)}) = \tilde{\zeta}_F(1-k, \delta_F^{(x)})$$

for all such x, since[6] $(\delta_{F^\sigma}^{(x^s)})_{g_{F^\sigma}} = \delta_{F^\sigma}^{(g_{F^\sigma}^{-1}x^s)} = \delta_{F^\sigma}^{((g_F^{-1}x)^s)}$ and $(\delta_F^{(x)})_{g_F} = \delta_F^{(g_F^{-1}x)}$. Viewing $\delta_F^{(x)}$ as a complex valued function on $G(F_S/F)/U$ and writing it as a $\mathbb{C}$-linear combination of the (abelian) characters χ of $G(F_S/F)/U$, it now suffices to check that $\tilde{\zeta}_F(1-k,\chi) = \tilde{\zeta}_{F^\sigma}(1-k,\chi^s)$, and this follows from the compatibility of the Artin L-functions with inflation and induction. Indeed, inflating χ from $G(F_S/F)/U$ to $G(F_S/F)$ and further to $G(L_S/F)$ and then inducing up to $G(L_S/K)$, and analogously with χ, F, U replaced by χ^s, F^σ, U^s (note that U^s is well-defined), we have

$$\mathrm{ind}_{G(L_S/F)}^{G(L_S/K)} \mathrm{infl}_{G(F_S/F)/U}^{G(L_S/F)}(\chi) = \mathrm{ind}_{G(L_S/F^\sigma)}^{G(L_S/K)} \mathrm{infl}_{G(F_S^s/F^\sigma)/U}^{G(L_S/F^\sigma)}(\chi^s) \ .$$

Lemma 3 is established. □

[6] recall our convention $\varepsilon_g(g') = \varepsilon(gg')$

Set $\Sigma_F = G(L/F)$, and, for any set X carrying a natural Σ-action, denote the stabilizer subgroup of $x \in X$ in Σ by $\mathrm{St}_\Sigma(x) = \{\sigma \in \Sigma : \sigma(x) = x\}$. Also, $\mathbb{Z}_{(p)} \subset \mathbb{Q}$ is the localization of $\mathbb{Z}$ at its prime ideal $p\mathbb{Z}$.

Proposition 4. *For $g \in G_S$ set $g_F = \mathrm{ver}_K^F g$ for all $K \subseteq F \subseteq L$. Then*

$$(4\star) \qquad \sum_{K \subseteq F \subseteq L} \mu_\Sigma(\Sigma_F)\tilde{\Delta}_{g_F}(1 - |\Sigma_F|k, \varepsilon_L \mathrm{ver}_F^L) \equiv 0 \mod |\mathrm{St}_\Sigma(\varepsilon_L)|\mathbb{Z}_{(p)},$$

for all even locally constant $\mathbb{Z}_{(p)}$-valued functions ε_L on H_S, implies that

$$\mathfrak{s}_g \stackrel{\text{def}}{=} \sum_F \mu_\Sigma(\Sigma_F)\mathrm{ver}_F^L(\tilde{\lambda}_{g_F})$$

has image, under the map $\mathbb{Z}_p[[H_S]] \to \mathbb{Z}_p[[H_S^+]]$, in $\mathrm{tr}_\Sigma(\mathbb{Z}_p[[H_S^+]]$.

For the proof of the proposition we first recall that a locally constant function ε_L on H_S is *even* if $\varepsilon_L(c_w h) = \varepsilon_L(h)$ for all $h \in H_S$ and all 'Frobenius elements' c_w at the archimedean primes w of L, i.e., at the restrictions $c_w \in H_S$ of complex conjugation with respect to the embeddings $L_S \hookrightarrow \mathbb{C}$ inducing w on L. We denote by C the group generated by the c_w's, so $H_S^+ = H_S/C$.

We next observe that $\mathfrak{s}_g \in \mathbb{Z}_p[[H_S]]^\Sigma$. Namely, $\mathrm{ver}_F^L(\tilde{\lambda}_{g_F})^s = \mathrm{ver}_{F^\sigma}^L(\tilde{\lambda}_{g_F}^s) = \mathrm{ver}_{F^\sigma}^L(\tilde{\lambda}_{g_{F^\sigma}})$, by (2) of Lemma 3. Moreover, $\mu_\Sigma(\Sigma_F) = \mu_\Sigma(\Sigma_{F^\sigma})$.

Turning finally to the image of $\mathfrak{s}_g$ under the map $\mathbb{Z}_p[[H_S]] \to \mathbb{Z}_p[[H_S^+]]$, we first replace the diagram in [RW7,p.718] by the diagram below, in which $N = \ker \mathrm{ver}_F^L$:

$$\begin{array}{ccc}
\mathbb{Z}_p[[G(F_S/F]] & \to & \varprojlim_{U \geq N} \mathbb{Z}_p[G(F_S/F)/U]/p^{m_F(U)} \\
\mathrm{ver}_F^L \downarrow & & \downarrow \\
\mathbb{Z}_p[[H_S]] & \stackrel{\simeq}{\to} & \varprojlim_{\Sigma-\text{stable}\,V} \mathbb{Z}_p[H_S/V]/p^{m_L(V)-e_F}
\end{array}.$$

Recall here that the right vertical map takes $(x_U)_U$ to $(y_V)_V$ by means of

$$\Xi : \quad \mathbb{Z}_p[G(F_S/F)/U]/p^{m_F(U)} \xrightarrow{\mathrm{ver}} \mathbb{Z}_p[H_S/V]/p^{m_F(U)} \to \mathbb{Z}_p[H_S/V]/p^{m_L(V)-e_F},$$

whenever $U \leq (\mathrm{ver}_F^L)^{-1}(V)$. [7]

Since the $m_L(V)$'s are unbounded, there are admissible open Σ-stable $V \leq H_S$ with $m_L(V) - e_F \geq e_K$ ($\forall\, F$). For any such V then $\mathbb{Z}_p[H_S/V]/p^{m_L(V)-e_F}$ maps onto $\mathbb{Z}_p[H_S/V]/|\Sigma|$ and we write the image of $\mathfrak{s}_g$ in here as $\sum_{y \in H_S/V} c_y y$. Because Σ fixes $\mathfrak{s}_g$, $c_{y^\sigma} = c_y$ for all σ. Since $\sum_{\sigma \in \Sigma \bmod \mathrm{St}_\Sigma(y)} c_{y^\sigma} y^\sigma = c_y \sum y^\sigma$, it follows that $\mathfrak{s}_g$ will be in $\mathrm{tr}_\Sigma(\mathbb{Z}_p[H_S/V]) + |\Sigma|\mathbb{Z}_p[H_S/V]$ provided that

$$c_y \equiv 0 \mod |\mathrm{St}_\Sigma(y)|.$$

[7] Note that 'ver' is the $\mathbb{Z}_p$-linear map induced by the group homomorphism obtained by factoring $G(F_S/F) \stackrel{\mathrm{ver}_F^L}{\to} H_S \to H_S/V$ through $G(F_S/F) \to G(F_S/F)/U$.

We compute the coefficient c_y of the image of $\mathfrak{s}_g$. By Proposition 2 with $U = (\mathrm{ver}_F^L)^{-1}(V)$ (compare [RW7, (ii) on p.719] noting that any k is allowed) $\mathrm{ver}_F^L(\tilde{\lambda}_{g_F})$ has image

$$\sum_{x \in G(F_S/F)/U} \tilde{\Delta}_{g_F}(1 - |\Sigma_F|k, \delta_F^{(x)})\mathcal{N}_F(x)^{-|\Sigma_F|k}\mathrm{ver}_F^L(x)\,,$$

where $\delta_F^{(x)}$ is the characteristic function of the coset $x \subseteq G(F_S/F)$.

Since $G(F_S/F) \overset{\mathrm{ver}_F^L}{\to} H_S \to H_S/V$ has kernel $U = (\mathrm{ver}_F^L)^{-1}(V)$, either y is not in the image of ver_F^L or $y = \mathrm{ver}_F^L(x^{(F)})$ for a unique $x^{(F)} \in G(F_S/F)/U$. Note that $\delta_L^{(y)}\mathrm{ver}_F^L$ is $= 0$ in the first case and $= \delta_F^{(x^{(F)})}$ in the second, when also $\mathcal{N}_F(x^{(F)})^{-|\Sigma_F|k} = \mathcal{N}_L(\mathrm{ver}_F^L(x^{(F)}))^{-k}$. Thus, Möbius-summing over F we obtain

$$c_y = (\sum_F \mu_\Sigma(\Sigma_F)\tilde{\Delta}_{g_F}(1 - |\Sigma_F|k, \delta_L^{(y)}\mathrm{ver}_F^L)\,)\mathcal{N}_L(y)^{-k}\,.$$

Our hypothesis (4$\star$) now implies that $\mathfrak{s}_g$ is in $\mathrm{tr}_\Sigma(\mathbb{Z}_p[H_S/V]) + |\Sigma|\mathbb{Z}_p[H_S/V]$ for all $V \geq C$ (recall that $\delta_L^{(y)}$ is even when $V \geq C$ and $y \in H_S/V$). Since $\mathfrak{s}_g$ is fixed by Σ and $|\Sigma|\mathbb{Z}_p[H_S/V]^\Sigma \subseteq \mathrm{tr}_\Sigma(\mathbb{Z}_p[H_S/V])$, it follows that $\mathfrak{s}_g \in \mathrm{tr}_\Sigma\mathbb{Z}_p[H_S/V]$.

This finishes the proof of Proposition 4. □

2. Applying the q-expansion principle of [DR]

Given an even integer k and an even locally constant $\mathbb{Z}_{(p)}$-valued function ε_L on H_S, choose an open subgroup V of H_S so that ε_L is constant on each coset H_S/V and let $\mathfrak{f} \subset |\Sigma|\mathfrak{o}_K$ be an integral ideal, with all its prime factors contained in S, so that, for all $K \subseteq F \subseteq L$, $\mathfrak{f}\mathfrak{o}_F$ is a multiple of the conductor of the field fixed by $(\mathrm{ver}_F^L)^{-1}(V)$ acting on F_S.

As in [DR,p.229] we write $\hat{K}$ for the ring of 'finite' adèles of K and let $j : \hat{K}^\times \to G_S\,, \psi : \hat{K}^\times \to \varprojlim_{\mathfrak{f}'} G_{\mathfrak{f}'}$, with $\mathfrak{f}'$ running over the multiples of $\mathfrak{f}$ that have all their prime divisors in S, be the maps defined in [DR,p.243].

Proposition 5. *There exist $\gamma \in \hat{K}^\times$ so that*

a) γ and γ^{-1} are in $1 + \hat{\mathfrak{f}}$, and

b) the image $g \in G_S$ of γ under $j : \hat{K}^\times \to G_S$ has $\mathrm{ver}_K^F(g) \in G(F_S/F)$ <u>not</u> in the kernel of $G(F_S/F) \to \Gamma_F$, for all $K \subseteq F \subseteq L$.

Proof. Choose $f > 0$ in $\mathbb{Z} \cap \mathfrak{f}$, and let $\alpha \in \hat{\mathbb{Q}}^\times$ be the 'finite' idèle with components $1 + f$ (respectively 1) at primes $q|f$ (respectively $q \nmid f$). Then α and α^{-1} belong to $1 + \widehat{\mathbb{Z}f}$. For every extension $\mathbb{Q} \hookrightarrow F$ let α_F be the image of α under the diagonal inclusion $\hat{\mathbb{Q}}^\times \hookrightarrow \hat{F}^\times$. We use $\varprojlim_{\mathfrak{f}'} G_{\mathfrak{f}'} = G_S$, on identifying the inverse limit of ray class groups with the one of the corresponding Galois groups, as is conventional in [DR,p.240]; so $j = \psi = (\psi_{\mathfrak{f}'})_{\mathfrak{f}'}$ by [DR,2.33].

Now $\psi_{\mathfrak{f}'}(\alpha_K) = \psi_{\mathfrak{f}'}((1+f)^{-1}\alpha_K)$, since $1+f \in K^{\gg 0}$, and $(1+f)^{-1}\alpha_K$ has $\mathfrak{q}$-component 1 (respectively $(1+f)^{-1}$) when $\mathfrak{q}|\mathfrak{f}$ (respectively $\mathfrak{q} \nmid \mathfrak{f}$), so that the (fractional) ideal of $\mathfrak{o} = \mathfrak{o}_K$ 'generated' by $(1+f)^{-1}\alpha_K$ is $\mathfrak{a} = (1+f)^{-1}\mathfrak{o}$, which is coprime to $\mathfrak{f}$.

Recall that K_S contains the $\mathbb{Z}_p$-cyclotomic extension K_∞ of K. By [Se, 2.2], $g = \psi(\alpha_K) \in G_S$ acts on p-power roots of unity as $g = (\mathfrak{a}, K_S/K)$, i.e., by raising them to the power $\mathcal{N}(\mathfrak{a}) = (1+f)^{-[K:\mathbb{Q}]}$. It follows that the image of g under $G_S \to \Gamma_K$ acts non-trivially, so *a)* holds for α_K and also *b)* when $K = F$.

The argument for F is the same with $K, \mathfrak{f}, S$ replaced by $F, \mathfrak{f}\mathfrak{o}_F, S_F$ and shows that $\psi_F(\alpha_F)$ acts on p-power roots of unity by $(1+f)^{-[F:\mathbb{Q}]}$. Note also that $\psi_F(\alpha_F) = \mathrm{ver}_K^F(g)$ by the usual relation between inclusion and transfer. Setting $\gamma = \alpha_K$ we then get *a)* and *b)*.

This completes the proof of Proposition 5. □

The next three results of [RW7] concern Hilbert modular forms with emphasis on their q-expansions. Thus Lemma 6 constructs a Hecke operator U_β on $\mathcal{M}_k(\Gamma_{00}(\mathfrak{f}), \mathbb{C})$, Lemma 7 discusses restriction $\mathrm{res}_F^K : \mathcal{M}_k(\Gamma_{00}(\mathfrak{f}\mathfrak{o}_F), \mathbb{C}) \to \mathcal{M}_{[F:K]k}(\Gamma_{00}(\mathfrak{f}), \mathbb{C})$ for field extensions F/K, and Proposition 8 is our bridge to [DR].

With k, ε_L and $\mathfrak{f}$ as at the beginning of the section and any $g \in G_S$, we next exhibit a Hilbert modular form $\mathcal{E}$ in $\mathcal{M}_{|\Sigma|k}(\Gamma_{00}(\mathfrak{f}), \mathbb{C})$ with the constant term of its standard q-expansion equal to $\sum_F \mu_\Sigma(\Sigma_F)\tilde{\Delta}_{g_F}(1-|\Sigma_F|k, \varepsilon_L \mathrm{ver}_F^L)$ (compare $(4\star)$).

First, by [DR, (6.1)], in the form of Proposition 8, there are modular forms

$$E_F \stackrel{\mathrm{def}}{=} G_{|\Sigma_F|k, \varepsilon_L \mathrm{ver}_F^L} \in \mathcal{M}_{|\Sigma_F|k}(\Gamma_{00}(\mathfrak{f}\mathfrak{o}_F), \mathbb{C})$$

of weight $|\Sigma_F|k$ with standard q-expansion

$$\tilde{\zeta}_F(1-|\Sigma_F|k, \varepsilon_L \mathrm{ver}_F^L) + \sum_{\substack{\mu \gg 0 \\ \mu \in \mathfrak{o}_F}} \Big(\sum_{\substack{\mu \in \mathfrak{a} \subseteq \mathfrak{o}_F \\ \mathfrak{a} \text{ prime to } S}} \varepsilon_L \mathrm{ver}_F^L(\mathfrak{a}) \mathcal{N}_F(\mathfrak{a})^{|\Sigma_F|k-1} \Big) q_F^\mu \ .$$

Appealing to Lemma 7 and Lemma 6, we apply res_F^K and the Hecke operator $U_{[F:K]}$ to the modular form E_F displayed above, and obtain, for each F, the new modular form

$$\mathcal{E}_F = (\mathrm{res}_F^K E_F)_{|[F:K]||\Sigma_F|k} U_{[F:K]}$$

of weight $|\Sigma|k$ in $\mathcal{M}_{|\Sigma|k}(\Gamma_{00}(\mathfrak{f}), \mathbb{C})$ and with standard q-expansion

$$\tilde{\zeta}_F(1-|\Sigma_F|k, \varepsilon_L \mathrm{ver}_F^L) + \sum_{\substack{\alpha \gg 0 \\ \alpha \in \mathfrak{o}_K}} \Big(\sum_{[\alpha]_F} \varepsilon_L(\mathfrak{a}_F \mathfrak{o}_L) \mathcal{N}_F(\mathfrak{a}_F)^{[L:F]k-1} \Big) q_K^\alpha \, ,$$

where, for $K \subseteq F \subseteq L$, $[\alpha]_F$ denotes the set of all pairs $(\alpha_F, \mathfrak{a}_F)$ satisfying

$$0 \ll \alpha_F \in \mathfrak{a}_F \subseteq \mathfrak{o}_F \, , \ \mathfrak{a}_F \text{ prime to } S \, , \ \mathrm{tr}_{F/K}(\alpha_F) = [F:K]\alpha \ .$$

Here we have used $(\varepsilon_L \circ \mathrm{ver}_F^L)(\mathfrak{a}_F) = (\varepsilon_L \mathrm{ver}_F^L)(\mathfrak{a}_F, F_S/F) = \varepsilon_L(\mathfrak{a}_F \mathfrak{o}_L)$.

We Möbius-sum all these $\mathcal{E}_F$ and arrive at the modular form

$$\mathcal{E} \stackrel{\text{def}}{=} \sum_{K\subseteq F\subseteq L} \mu_\Sigma(\Sigma_F)\mathcal{E}_F$$

in $\mathcal{M}_{|\Sigma|k}(\Gamma_{00}(\mathfrak{f}),\mathbb{C})$ whose standard q-expansion has constant coefficient

$$(0) \qquad \textstyle\sum_F \mu_\Sigma(\Sigma_F)\tilde{\zeta}_F(1-|\Sigma_F|k, \varepsilon_L \mathrm{ver}_F^L)$$

and higher coefficients

$$(\alpha) \qquad \textstyle\sum_F \mu_\Sigma(\Sigma_F)\sum_{[\alpha]_F} \varepsilon_L(\mathfrak{a}_F\mathfrak{o}_L)\mathcal{N}_F(\mathfrak{a}_F)^{[L:F]k-1}\,,$$

at $0 \ll \alpha \in \mathfrak{o}_K$.

Proposition 9. *Assume that k is an even positive integer and ε_L an even locally constant $\mathbb{Z}_p$-valued function on H_S. Then*

$$\sum_{K\subseteq F\subseteq L} \mu_\Sigma(\Sigma_F)\sum_{[\alpha]_F} \varepsilon_L(\mathfrak{a}_F\mathfrak{o}_L)\mathcal{N}_F(\mathfrak{a}_F)^{[L:F]k-1} \equiv 0 \mod |\mathrm{St}_\Sigma(\varepsilon_L)|\mathbb{Z}_{(p)}\ .$$

Proof. Utilizing the natural action of Σ on the set

$$[\alpha]_L = \{(\alpha_L,\mathfrak{a}_L) : 0 \ll \alpha_L \in \mathfrak{a}_L \subseteq \mathfrak{o}_L\,,\ \mathfrak{a}_L \text{ prime to } S\,,\ \mathrm{tr}_{L/K}(\alpha_L) = [L:K]\alpha\}\,,$$

given by $(\alpha_L,\mathfrak{a}_L)^\sigma = (\alpha_L^\sigma,\mathfrak{a}_L^\sigma)$, we identify the set $[\alpha]_F$ with the subset $[\alpha]_L^{\Sigma_F}$ of Σ_F-fixed points in $[\alpha]_L$ by means of the map

$$\iota_F : [\alpha]_F \to [\alpha]_L\,, \quad (\alpha_F,\mathfrak{a}_F) \mapsto (\alpha_F,\mathfrak{a}_F\mathfrak{o}_L)\ .$$

Indeed, ι_F is obviously injective and has image $[\alpha]_L^{\Sigma_F}$ because

$\alpha_L \in F$ if $\alpha_L^\sigma = \alpha_L$ for all $\sigma \in \Sigma_F$,

$\mathfrak{a}_L = \mathfrak{a}_L^\sigma$ for all $\sigma \in \Sigma_F$ implies $\mathfrak{a}_L = \mathfrak{a}_F\mathfrak{o}_L$ for some integral ideal $\mathfrak{a}_F$ of F, since $\mathfrak{a}_L$ is prime to S and whence every prime divisor of $\mathfrak{a}_L$ is unramified in L/K .

As a first consequence, formula (α) can be rewritten as

$$(\alpha') \qquad \sum_{K\subseteq F\subseteq L} \mu_\Sigma(\Sigma_F) \sum_{(\beta,\mathfrak{b})\in[\alpha]_L^{\Sigma_F}} \varepsilon_L(\mathfrak{b})\mathcal{N}_L(\mathfrak{b})^{k-\frac{1}{|\Sigma_F|}}\,,$$

as $\mathcal{N}_L(\mathfrak{a}_F\mathfrak{o}_L) = \mathcal{N}_F(\mathfrak{a}_F)^{[L:F]}$.

We isolate the part of the sum (α') that belongs to a fixed $(\beta,\mathfrak{b}) \in [\alpha]_L$. It is

$$\sum_{F \text{ so } \Sigma_F \le \mathrm{St}_\Sigma(\beta,\mathfrak{b})} \mu_\Sigma(\Sigma_F)\varepsilon_L(\mathfrak{b})\mathcal{N}_L(\mathfrak{b})^{k-\frac{1}{|\Sigma_F|}}\,,$$

and correspondingly, with $(\beta,\mathfrak{b})$ replaced by $(\beta,\mathfrak{b})^\sigma$,

$$\sum_{F \text{ so } \Sigma_F \le \mathrm{St}_\Sigma(\beta,\mathfrak{b})^\sigma} \mu_\Sigma(\Sigma_F)\varepsilon_L(\mathfrak{b}^\sigma)\mathcal{N}_L(\mathfrak{b}^\sigma)^{k-\frac{1}{|\Sigma_F|}}\ .$$

The group Σ acts on H_S and so on ε_L by ${\varepsilon_L}^\sigma(h) = \varepsilon_L(h^{\sigma^{-1}})$. We now consider the part of the sum (α') that belongs to the $\mathrm{St}_\Sigma(\varepsilon_L)$-orbit of $(\beta, \mathfrak{b})$:

$$\text{(i)} \qquad \sum_{\sigma \in [\mathrm{St}_\Sigma(\varepsilon_L) \cap \mathrm{St}_\Sigma(\beta,\mathfrak{b}) \backslash \mathrm{St}_\Sigma(\varepsilon_L)]} \varepsilon_L(\mathfrak{b}) \mathcal{N}_L(\mathfrak{b})^k \sum_{F \text{ so } \Sigma_F \le \mathrm{St}_\Sigma(\beta,\mathfrak{b})^\sigma} \mu_\Sigma(\Sigma_F) \mathcal{N}_L(\mathfrak{b})^{-\frac{1}{|\Sigma_F|}} \ .$$

Here, $[\mathrm{St}_\Sigma(\varepsilon_L) \cap \mathrm{St}_\Sigma(\beta,\mathfrak{b}) \backslash \mathrm{St}_\Sigma(\varepsilon_L)]$ is a set of right coset representatives of $\mathrm{St}_\Sigma(\varepsilon_L) \cap \mathrm{St}_\Sigma(\beta,\mathfrak{b})$ in $\mathrm{St}_\Sigma(\varepsilon_L)$. Note that the sum (α') is the sum of all such orbit sums (i).

Because of

$\Sigma_F \le \mathrm{St}_\Sigma(\beta,\mathfrak{b})^\sigma \iff \Sigma_{F^{\sigma^{-1}}} \le \mathrm{St}_\Sigma(\beta,\mathfrak{b})$ (as $\mathrm{St}_\Sigma(\beta,\mathfrak{b})^\sigma = (\mathrm{St}_\Sigma(\beta,\mathfrak{b}))^\sigma$ and $\Sigma_F^{\sigma^{-1}} = \Sigma_{F^{\sigma^{-1}}}$),

$\mu_\Sigma(\Sigma_F) = \mu_\Sigma(\Sigma_{F^{\sigma^{-1}}})$ (a direct consequence of the definition of the Möbius function),

and $|\Sigma_F| = |\Sigma_{F^{\sigma^{-1}}}|$,

the inner sums of (i) are independent of σ. Hence, if we can show

$$\text{(ii)} \qquad \textstyle\sum_{F \text{ so } \Sigma_F \le \mathrm{St}_\Sigma(\beta,\mathfrak{b})} \mu_\Sigma(\Sigma_F) \mathcal{N}_L(\mathfrak{b})^{-\frac{1}{|\Sigma_F|}} \equiv 0 \mod |\mathrm{St}_\Sigma(\varepsilon_L) \cap \mathrm{St}_\Sigma(\beta,\mathfrak{b})| \mathbb{Z}_{(p)} \,,$$

then sum (i) is $\equiv 0 \mod |\mathrm{St}_\Sigma(\varepsilon_L)|\mathbb{Z}_{(p)}$ and the proof of the proposition will be complete.

For (ii), we first shorten the notation by setting $P = \mathrm{St}_\Sigma(\beta,\mathfrak{b}) \le \Sigma$ and $r = \mathcal{N}_L(\mathfrak{b})^{-\frac{1}{|\mathrm{St}_\Sigma(\beta,\mathfrak{b})|}}$ which is a unit in $\mathbb{Z}_{(p)}$ because $\mathfrak{b} = \mathfrak{a}_M \mathfrak{o}_L$ if $\Sigma_M = P$ (compare with the proof of (α') above). This turns the left hand side of (ii) into $\sum_{1 \le P' \le P} \mu_P(P') r^{[P:P']}$, as obviously $\mu_\Sigma(P') = \mu_P(P')$. Applying now the Claim below we obtain

$$\sum_{1 \le P' \le P} \mu_P(P') r^{[P:P']} \equiv 0 \mod |P|\mathbb{Z}_{(p)} \,,$$

which is even stronger than (ii).

Claim: *Let P be a finite p-group and r a unit in $\mathbb{Z}_{(p)}$. Then*

$$\sum_{1 \le P' \le P} \mu_P(P') r^{[P:P']} \equiv 0 \mod |P|\mathbb{Z}_{(p)} \ .$$

To see this, let $z \in \mathbb{Z}_p$ satisfy $z^{p-1} = 1$ and $r \equiv z \mod p$, hence

$$r^{p^n} \equiv z^{p^n} = z \mod p^{n+1}\,, \quad \text{for } n \ge 0\,.$$

From [HIÖ,p.717] we obtain $|P'| \,\Big|\, p\mu_{P'}(P')$, for $P' \le P$. Therefore, and as $\mu_P(P') = \mu_{P'}(P')$, $\mu_P(P') r^{[P:P']} \equiv \mu_P(P') z \mod |P|\mathbb{Z}_p$. Consequently,

$$\sum_{1 \le P' \le P} \mu_P(P') r^{[P:P']} \equiv z \cdot \sum_{1 \le P' \le P} \mu_P(P') = 0 \mod |P|\mathbb{Z}_p \ ,$$

as $\sum_{1 \le P' \le P} \mu_P(P') = 0$, by the definition of the Möbius function.

Since $\mathbb{Q} \cap \mathbb{Z}_p = \mathbb{Z}_{(p)}$, this proves the Claim and also ends the proof of Proposition 9. $\square$

Choosing $\gamma \in \hat{K}^\times$ with $j(\gamma) = g$ and denoting by $\mathcal{E}_\gamma$ the q-expansion of $\mathcal{E}$ at the cusp determined by γ, set $\mathcal{E}(\gamma) \overset{\mathrm{def}}{=} \mathcal{N}_K(\gamma_p)^{-|\Sigma|k}\mathcal{E}_\gamma$ (so $\mathcal{E}(1)$ is the standard q-expansion of $\mathcal{E}$). Then [DR, (0.3)] implies that

> *the constant term of* $\mathcal{E}(1) - \mathcal{E}(\gamma)$ *is contained in* $p^n\mathbb{Z}_{(p)}$, *provided that* $\mathcal{E}(1)$ *has all non-constant coefficients contained in* $p^n\mathbb{Z}_{(p)}$,

and, by Proposition 9, this applies with $p^n = |\mathrm{St}_\Sigma(\varepsilon_L)|$, so the constant term of $\mathcal{E}(1) - \mathcal{E}(\gamma)$ is in $|\mathrm{St}_\Sigma(\varepsilon_L)|\mathbb{Z}_{(p)}$.

3. Conclusion of the proof : special cusps

By what has been said at the end of the previous section we need to compute the constant coefficients of $\mathcal{E}(\gamma)$ and $\mathcal{E}(1) - \mathcal{E}(\gamma)$. We do this for the γ's of Proposition 5 and then prove the Theorem stated in the introduction.

Lemma 10. *Setting* $g = j(\gamma)$, *with* γ *as in Proposition 5, the constant term of* $\mathcal{E}(\gamma)$ *is*

$$\mathcal{N}_K(g)^{|\Sigma|k} \cdot \Big(\sum_F \mu_\Sigma(\Sigma_F)\tilde{\zeta}_F(1 - |\Sigma_F|k, (\varepsilon_L \mathrm{ver}_F^L)_{g_F}) \Big) .$$

For the proof we first show that $\mathrm{res}_F^K E_F$ has constant term $\mathcal{N}_K((\gamma))^{|\Sigma|k}\tilde{\zeta}_F(1-|\Sigma_F|k,(\varepsilon_L\mathrm{ver}_F^L)_{g_F})$ at the cusp determined by $\gamma \in \hat{K}^\times$. By (2) of Lemma 7, this constant term of $\mathrm{res}_F^K E_F$ is equal to the one of E_F at the cusp determined by $\gamma \in \hat{F}^\times$, whence, by (2) of Proposition 8, equals

$$\mathcal{N}_F((\gamma)_F)^{|\Sigma_F|k}\tilde{\zeta}_F(1 - |\Sigma_F|k, (\varepsilon_L \mathrm{ver}_F^L)_{g_F}) = \mathcal{N}_K((\gamma))^{|\Sigma|k}\tilde{\zeta}_F(1 - |\Sigma_F|k, (\varepsilon_L \mathrm{ver}_F^L)_{g_F}) ,$$

because $(\gamma)_F = (\gamma)_K\mathfrak{o}_F$.

We next check that $(\mathrm{res}_F^K E_F)_{||\Sigma|k}U_{[F:K]}$ and $\mathrm{res}_F^K E_F$ have the same constant term at the cusp determined by $\gamma \in \hat{K}^\times$. By *a)* of Proposition 5, $M \overset{\mathrm{def}}{=} \left(\begin{smallmatrix} \gamma & 0 \\ 0 & \gamma^{-1} \end{smallmatrix}\right)$ is in $\widehat{\Gamma_{00}(\mathfrak{f})}$, so $M = M_1M_2$ with $M_2 = I$ in the notation of [DR,p.262], hence, by [DR,5.8], the constant terms referred to above are the constant terms of the standard q-expansions of $((\mathrm{res}_F^K)_{||\Sigma|k}U_{[F:K]})|M_2 = (\mathrm{res}_F^K)_{||\Sigma|k}U_{[F:K]}$ and $(\mathrm{res}_F^K E_F)|M_2 = \mathrm{res}_F^K E_F$, which agree by Lemma 6.

Möbius-summing, we have shown that $\mathcal{E}_\gamma$ has constant term

$$\mathcal{N}_K((\gamma))^{|\Sigma|k} \sum_F \mu_\Sigma(\Sigma_F)\tilde{\zeta}_F(1 - |\Sigma_F|k, (\varepsilon_L \mathrm{ver}_F^L)_{g_F}) ,$$

hence $\mathcal{E}(\gamma)$ has the required constant term, because $\mathcal{N}_K(\gamma_p)^{-1}\mathcal{N}_K((\gamma)) = \mathcal{N}_{K,p}(g)$, by (3) of Proposition 8.

This completes the proof of Lemma 10. □

We can now complete the proof of the Theorem. It follows from equation (0) and Lemma 10 that for such γ the constant term of $\mathcal{E}(1)-\mathcal{E}(\gamma)$ is

$$\begin{aligned}&\textstyle\sum_F \mu_\Sigma(\Sigma_F)[\tilde\zeta_F(1-|\Sigma_F|k,\varepsilon_L\mathrm{ver}_F^L)-\mathcal{N}_K(g)^{|\Sigma|k}\tilde\zeta_F(1-|\Sigma_F|k,(\varepsilon_L\mathrm{ver}_F^L)_{g_F})]\\ &=\textstyle\sum_F \mu_\Sigma(\Sigma_F)\tilde\Delta_{g_F}(1-|\Sigma_F|k,\varepsilon_L\mathrm{ver}_F^L)\ ;\end{aligned}$$

the latter since $\mathcal{N}_F(\mathrm{ver}_K^F g)=\mathcal{N}_K(g)^{[F:K]}$.

Note at this stage that it is this sum, $\sum_F \mu_\Sigma(\Sigma_F)\tilde\Delta_{g_F}(1-|\Sigma_F|k,\varepsilon_L\mathrm{ver}_F^L)$, which is referred to in ($4\star$). The last sentence of §2 now implies that this sum is $\equiv 0 \mod |\mathrm{St}_\Sigma(\varepsilon_L)|\mathbb{Z}_{(p)}$, for every even locally constant ε_L, thus verifying the hypothesis of Proposition 4.

The Theorem is now the conclusion of Proposition 4. □

Remark. The application of the Theorem to equivariant Iwasawa theory, in [RW10], is different from that proposed in [RW7, pp.715/716] since the specialization map $G(L_S/K)\to G(\mathbf{K}/K)$ extends to a map of localizations in the sense of [loc.cit.] only when $G(\mathbf{K}/K)$ is a pro-p group.

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